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A method of coupling non-linear hyperbolic systems: examples in CFD and plasma physics

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SUMMARY

This paper analyses a method of coupling systems of conservation laws with examples in two fluid flows. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: conservation laws; boundary value problems; numerical methods

1. INTRODUCTION

In the modelling of complex problems, different mathematical models are frequently used in different regions of interest. For instance, one can take into account some physical effects (multi-dimensional effects, compressibility, temperature variations, Joule effect in the following example, etc.) in some domain where they are supposed to be important or assume that they are negligible elsewhere which amounts to drop the corresponding terms in the equations of the complete model. We have studied an example of this situation in the context of plasma fluids with two different current densities. Also different closure relations may be considered. A simple example of this last situation which we analyse below is provided by the flow of two perfect fluids with different equations of state separated by a moving interface. This leads to couple different systems and we consider a coupling procedure which was initially motivated by numerical considerations (see References [1, 2]).

We are thus interested in the coupling of non-linear hyperbolic systems of conservation laws at an interface which we assume, for simplicity, is fixed. In the scalar case, the complete study from both mathematical and numerical points of view was considered in Reference [3]. Here, we take up the coupling of systems and the situation is far more complicated although we restrict ourselves to the one-dimensional case and to systems of the same size. Only in the

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linear case is it possible to justify completely the analysis of the continuous problem. We can also study coupled Riemann problem corresponding to an initial data constant on each side of the interface. Though simple in their formulation, they already provide interesting features.

2. DEFINITION OF THE COUPLED PROBLEM

Let $\Omega \subset \mathbb{R}^d$ be the set of states and let $\mathbf{f}_\alpha, \alpha = L, R$, be two 'smooth' functions from Ω into $\mathbb{R}^d$. Given a function $\mathbf{u}_0 : x \in \mathbb{R} \to \mathbf{u}_0(x)$, we want to find a function $\mathbf{u} : (x,t) \in \mathbb{R} \times \mathbb{R}_+ \to \mathbf{u}(x,t) \in \Omega$, satisfying

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}_{\mathrm{L}}(\mathbf{u}) = \mathbf{0}, \quad x < 0, \ t > 0 \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}_{\mathrm{R}}(\mathbf{u}) = \mathbf{0}, \quad x > 0, \ t > 0 \tag{2}$$

and the initial condition

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad x \in \mathbb{R} \tag{3}$$

We assume that systems (1) and (2) are hyperbolic, i.e. for $\alpha = L, R$, the Jacobian matrix $A_\alpha(\mathbf{u}) \equiv \mathbf{f}'_\alpha(\mathbf{u})$ of $\mathbf{f}_\alpha(\mathbf{u})$ is diagonizable with real eigenvalues $\lambda_{\alpha,k}(\mathbf{u})$ and corresponding eigenvectors $\mathbf{r}_{\alpha,k}(\mathbf{u}), 1 \leqslant k \leqslant d$. At the interface $x = 0$, Equations (1)–(3) are supplemented with coupling conditions. These conditions are determined in order to obtain *two well-posed boundary value problems* on each side of the interface. Given a boundary state $\mathbf{b}$, an admissible boundary condition at $x = 0$, for instance for system (2), is defined following Reference [4] through a condition of the form $\mathbf{u}(0+,t) \in \mathcal{O}_{\mathrm{R}}(\mathbf{b}(t)), t > 0$, where the set $\mathcal{O}_{\mathrm{R}}$ contains the possible traces on $x = 0$ of the solutions of Riemann problems for system (2) between the left state $\mathbf{b}$ and any right state. There are other more fully justified definitions of boundary sets $\mathcal{O}(\mathbf{b})$ (see, Reference [5]); however, this condition is convenient for practical purposes. For the coupled system we thus require

$$\mathbf{u}(0-,t) \in \mathcal{O}_{\mathrm{L}}(\mathbf{u}(0+,t)) \quad \text{and} \quad \mathbf{u}(0+,t) \in \mathcal{O}_{\mathrm{R}}(\mathbf{u}(0-,t)) \tag{4}$$

(see, Reference [6] for details) with obvious notation for the boundary set $\mathcal{O}_{\mathrm{L}}$ corresponding to (1).

This coupling procedure was first introduced from numerical considerations. Indeed, a numerical coupling procedure was used by Abgrall and Karni [1] or Pougeard-Dulimbert [2], and in Reference [3] we proved that this lead in the scalar case to this continuous coupling condition.

Let us now precise the numerical approximation of problem (1)–(3) and (4). We use a finite volume scheme: given a uniform mesh space Δx and a time step Δt, we set $\mu = \Delta t/\Delta x$, $x_{j+1/2} = \left(j + \frac{1}{2}\right)\Delta x$, $j \in \mathbb{Z}$, $t_n = n\Delta t$, $n \in \mathbb{N}$, and $\mathbf{u}^0_{j+1/2} = 1/\Delta x \int_{x_j}^{x_{j+1}} \mathbf{u}_0(x)\,\mathrm{d}x$, $j \in \mathbb{Z}$. Then, for $\alpha = L, R$, we are given a numerical flux function $\mathbf{g}_\alpha : \Omega^2 \to \mathbb{R}^d$ consistent with $\mathbf{f}_\alpha$ and we consider the three-point numerical scheme

$$\mathbf{u}^{n+1}_{j-1/2} = \mathbf{u}^n_{j-1/2} - \mu(\mathbf{g}^n_{\mathrm{L},j} - \mathbf{g}^n_{\mathrm{L},j-1}), \quad j \leqslant 0 \tag{5}$$

$$\mathbf{u}^{n+1}_{j+1/2} = \mathbf{u}^n_{j+1/2} - \mu(\mathbf{g}^n_{\mathrm{R},j+1} - \mathbf{g}^n_{\mathrm{R},j}), \quad j \geqslant 0 \tag{6}$$

where $\mathbf{g}^n_{\alpha,j} = \mathbf{g}_\alpha(\mathbf{u}^n_{j-1/2}, \mathbf{u}^n_{j+1/2})$, $\alpha = \mathrm{L}, \mathrm{R}$. The coupling of the difference schemes is performed through the evaluation of $\mathbf{g}^n_{\alpha,0} = \mathbf{g}_\alpha(\mathbf{u}^n_{-1/2}, \mathbf{u}^n_{1/2})$, $\alpha = \mathrm{L}, \mathrm{R}$. Since in general $\mathbf{g}^n_{\mathrm{L},0} \neq \mathbf{g}^n_{\mathrm{R},0}$, the overall numerical scheme (5), (6) is not conservative. We proved that in most significant scalar situations, the scheme converges and that the limit solution satisfies (1)–(4). Note that (4) leads in many cases to the continuity at the interface $\mathbf{u}(0-,t) = \mathbf{u}(0+,t)$.

We now develop a detailed analysis of the coupling in the abstract linear case with constant coefficients.

3. THE COUPLING OF LINEAR SYSTEMS

Assume throughout this section

$$\mathbf{f}_\alpha(\mathbf{u}) = \mathbf{A}_\alpha \mathbf{u} \tag{7}$$

where the matrix $\mathbf{A}_\alpha$ is a real $d \times d$ matrix. The well-posedness of the coupled Cauchy problem depends on the number of entering or outgoing characteristic lines on each side of the interface; this problem may be well- or ill-posed (in the sense that it possesses a continuum of solutions). We suppose that the eigenvalues $\lambda_{\alpha,k}$ (associated with the eigenvectors $\mathbf{r}_{\alpha,k}$) of the matrix $\mathbf{A}_\alpha$ are real, distinct and ordered as

$$\lambda_{\mathrm{L},1} < \lambda_{\mathrm{L},2} < \cdots < \lambda_{\mathrm{L},q_\mathrm{L}} < 0 \leqslant \lambda_{\mathrm{L},q_\mathrm{L}+1} < \cdots < \lambda_{\mathrm{L},d} \tag{8}$$

$$\lambda_{\mathrm{R},1} < \lambda_{\mathrm{R},2} < \cdots < \lambda_{\mathrm{R},q_\mathrm{R}} \leqslant 0 < \lambda_{\mathrm{R},q_\mathrm{R}+1} < \cdots < \lambda_{\mathrm{R},d} \tag{9}$$

We denote by $\mathbf{l}_{\alpha,k}$ the corresponding eigenvectors of $\mathbf{A}^\mathrm{T}_\alpha$ and asume the normalization $\mathbf{l}^\mathrm{T}_{\alpha,j} \cdot \mathbf{r}_{\alpha,k} = \delta_{jk}$, $1 \leqslant j,k \leqslant d$. The description of set $\mathcal{O}_\alpha(\mathbf{b})$ is classical (see Reference [7] for instance) and we obtain by interpreting (4) that any solution of the coupled linear problem satisfies

$$\mathbf{l}^\mathrm{T}_{\mathrm{L},k} \cdot \mathbf{u}(0-,t) = \begin{cases} \mathbf{l}^\mathrm{T}_{\mathrm{L},k} \cdot \mathbf{u}(0+,t), & 1 \leqslant k \leqslant q_\mathrm{L} \\ \mathbf{l}^\mathrm{T}_{\mathrm{L},k} \cdot \mathbf{u}_0(-\lambda_{\mathrm{L},k} t), & q_\mathrm{L} + 1 \leqslant k \leqslant d \end{cases} \tag{10}$$

and

$$\mathbf{l}^\mathrm{T}_{\mathrm{R},k} \cdot \mathbf{u}(0+,t) = \begin{cases} \mathbf{l}^\mathrm{T}_{\mathrm{R},k} \cdot \mathbf{u}_0(-\lambda_{\mathrm{R},k} t), & 1 \leqslant k \leqslant q_\mathrm{R} \\ \mathbf{l}^\mathrm{T}_{\mathrm{R},k} \cdot \mathbf{u}(0-,t), & q_\mathrm{R} + 1 \leqslant k \leqslant d \end{cases} \tag{11}$$

Since with any solution $\mathbf{u}(0\pm,t)$ of (10)–(11), one can associate in a unique way a solution $\mathbf{u}$ of the coupled problem—indeed the boundary values together with the initial data $\mathbf{u}_0$ allow us to solve separately the initial boundary value problems for $x<0$ and $x>0$—the linear coupled problem has a unique solution if and only if system (10), (11) of 2d linear equations in the 2d unknown components of $\mathbf{u}(0\pm,t)$ has a unique solution. This leads us to consider three cases, depending on the sign of $q_\mathrm{L} - q_\mathrm{R}$.

Case (i): $q_\mathrm{L} = q_\mathrm{R}$. It is the simplest case and we can prove easily

Theorem 1

In the case (7), assuming (8), (9) and

$$q_\mathrm{L} = q_\mathrm{R} = q$$

the coupled problem (1)–(4), has a unique solution if and only if the sets $\{\mathbf{l}_{\mathrm{L},1},\dots,\mathbf{l}_{\mathrm{L},q},\mathbf{l}_{\mathrm{R},q+1},\dots,\mathbf{l}_{\mathrm{R},d}\}$, $\{\mathbf{l}_{\mathrm{R},1},\dots,\mathbf{l}_{\mathrm{R},q},\mathbf{l}_{\mathrm{L},q+1},\dots,\mathbf{l}_{\mathrm{L},d}\}$ are two bases of $\mathbb{R}^d$. The coupling conditions (10), (11) then yield the continuity of $\mathbf{u}$ at the interface

$$\mathbf{u}(0+,t)=\mathbf{u}(0-,t) \tag{12}$$

Case (ii): $q_{\mathrm{L}}<q_{\mathrm{R}}$. Let us set

$$q=q_{\mathrm{R}}=q_{\mathrm{L}}+m, \quad m\geqslant 1 \tag{13}$$

and introduce the space $E=[\mathbf{r}_{\mathrm{R},1},\dots,\mathbf{r}_{\mathrm{R},q}]\cap[\mathbf{r}_{\mathrm{L},q-m+1},\dots,\mathbf{r}_{\mathrm{L},d}]$. We shall assume

$$\dim E=m \tag{14}$$

The coupling conditions (10), (11) require that the 'jump' $\mathbf{v}$ at the interface be in E, where

$$\mathbf{v}(t)=\mathbf{u}(0+,t)-\mathbf{u}(0-,t) \tag{15}$$

The coupling problem has a unique solution if we can determine uniquely $\mathbf{v}$ and say $\mathbf{u}(0-,t)$, satisfying

$$\mathbf{l}_{\mathrm{L},k}^{\mathrm{T}}\cdot\mathbf{u}(0-,t) = \mathbf{l}_{\mathrm{L},k}^{\mathrm{T}}\cdot\mathbf{u}_0(-\lambda_{\mathrm{L},k}t), \quad q_{\mathrm{L}}+1\leqslant k\leqslant d \tag{16}$$

$$\mathbf{l}_{\mathrm{R},k}^{\mathrm{T}}\cdot\mathbf{u}(0-,t)+\mathbf{l}_{\mathrm{R},k}^{\mathrm{T}}\cdot\mathbf{v}(t) = \mathbf{l}_{\mathrm{R},k}^{\mathrm{T}}\cdot\mathbf{u}_0(-\lambda_{\mathrm{R},k}t), \quad 1\leqslant k\leqslant q_{\mathrm{R}} \tag{17}$$

Defining the space $F=\{\mathbf{l}\in[\mathbf{l}_{\mathrm{R},1},\dots,\mathbf{l}_{\mathrm{R},q}];\ \mathbf{l}^{\mathrm{T}}\cdot\mathbf{r}_{\mathrm{L},i}=0,\ 1\leqslant i\leqslant q-m\}$, and letting E° denote the orthogonal of E in $\mathbb{R}^d$, we have

Theorem 2
In the linear case (7), assuming (8), (9), (13) and (14) with

$$\text{the } p \text{ vectors } \mathbf{l}_{\mathrm{R},1},\dots,\mathbf{l}_{\mathrm{R},q-m},\mathbf{l}_{\mathrm{L},q-m+1},\dots,\mathbf{l}_{\mathrm{L},d} \text{ are linearly independant} \tag{18}$$

the coupled problem (1)–(3) and (4) has a unique solution if and only if the subspaces E and F satisfy $E^{\circ}\cap F=\{0\}$.

Proof
Condition (14) is equivalent to state that $\mathbf{l}_{\mathrm{L},1},\dots,\mathbf{l}_{\mathrm{L},q-m},\mathbf{l}_{\mathrm{R},q+1},\dots,\mathbf{l}_{\mathrm{R},d}$ are linearly independent, which is a necessary condition to solve (16), (17). Hypothesis (18) ensures that $\dim F=m$. Then some computations show that (17) resumes to a linear system in the m components of $\mathbf{v}$ in E of the form

$$\mathbf{l}_i^{\mathrm{T}}\cdot\mathbf{v}(t)=\text{known right-hand side},\ q-m+1\leqslant i\leqslant q$$

where the $\mathbf{l}_i$'s are a basis of F. This system is uniquely solvable if $E^{\circ}\cap F=\{0\}$. □

This result provides an easy way to check that the coupled problem is well-posed. We shall illustrate this case in the next section.

Case (iii): $q_{\mathrm{L}}<q_{\mathrm{R}}$ is easily handled with and we can state

Theorem 3
In the linear case (7), assume (8), (9) and

$$q_{\rm L} > q_{\rm R} \tag{19}$$

and, moreover, that the $d + q_{\rm L} - q_{\rm R}$ vectors $\mathbf{l}_{{\rm L},1}, \dots, \mathbf{l}_{{\rm L},q_{\rm L}}, \mathbf{l}_{{\rm R},q_{\rm R}+1}, \dots, \mathbf{l}_{{\rm R},d}$ span $\mathbb{R}^d$ and that the $d - (q_{\rm L} - q_{\rm R})$ vectors $\mathbf{l}_{{\rm R},1}, \dots, \mathbf{l}_{{\rm R},q_{\rm R}}, \mathbf{l}_{{\rm L},q_{\rm L}+1}, \dots, \mathbf{l}_{{\rm L},d}$ are linearly independent. Then the solutions of the coupled problem (1), (2), (3) and (4) form an affine variety of dimension $q_{\rm L} - q_{\rm R}$. Any solution $\mathbf{u}$ satisfies (12). Moreover, such a solution is uniquely determined if, in addition, the values of $\mathbf{l}_k^{\rm T} \cdot \mathbf{u}(0,t)$, $q_{\rm R} + 1 \leqslant k \leqslant q_{\rm L}$, are prescribed at the interface $x = 0$.

4. TWO FLUID FLOW (IN LAGRANGIAN CO-ORDINATES)

Consider the system of gas dynamics in Lagrangian co-ordinates

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}) = \mathbf{0}, \quad \mathbf{u} = (\tau, v, e)^{\rm T}, \quad \mathbf{f}(\mathbf{u}) = (-v, p, pv)^{\rm T} \tag{20}$$

In (20), x stands for a mass variable and τ denotes the specific volume, v the velocity, $e = \varepsilon + \frac{1}{2} v^2$ the specific total energy, ε the specific internal energy, and $p = p(\tau, \varepsilon)$. We study the coupling of two such systems at a contact discontinuity located at $x = 0$ and separating two fluids with different equations of state $p = p_\alpha(\tau, \varepsilon)$, $\alpha = {\rm L}, {\rm R}$. For instance for two ideal gases $p = (\gamma - 1)\varepsilon/\tau$ with $\gamma = \gamma_{\rm L}$ or $\gamma_{\rm R}$. Let $\mathbf{f}_\alpha(\mathbf{u})$, $\alpha = {\rm L}, {\rm R}$, denote the corresponding fluxes.

We begin by considering the *linearized case*, i.e. the coupling of two gas dynamics system linearized at two constant states $\mathbf{u}_{\rm L}$ and $\mathbf{u}_{\rm R}$ separated by a material discontinuity which thus satisfy the continuity of the velocity and pressure

$$v_{\rm L} = v_{\rm R}, \quad p_{\rm L}(\tau_{\rm L}, \varepsilon_{\rm L}) = p_{\rm R}(\tau_{\rm R}, \varepsilon_{\rm R}) \tag{21}$$

We have (7) with $\mathbf{A}_\alpha = \mathbf{A}_\alpha(\mathbf{u}_\alpha)$, $\alpha = {\rm L}, {\rm R}$, where $\mathbf{A}(\mathbf{u})$ is the Jacobian matrix of $\mathbf{f}(\mathbf{u})$ given (using the usual notations $p_\varepsilon = \partial p / \partial \varepsilon(\tau, \varepsilon)$, $p_\tau = \partial p / \partial \tau(\tau, \varepsilon)$) by

$$\mathbf{A}(\mathbf{u}) = \begin{pmatrix} 0 & -1 & 0 \\ p_\tau & -v p_\varepsilon & p_\varepsilon \\ v p_\tau & p - v^2 p_\varepsilon & v p_\varepsilon \end{pmatrix}$$

Here $d = 3$ and $\lambda_1 = -C < \lambda_2 = 0 < \lambda_3 = C$ where $C = \sqrt{-p_\tau + p p_\varepsilon}$ denotes the Lagrangian sound speed. Recall that the (right) eigenvectors of $\mathbf{A}(\mathbf{u})$ can be chosen as

$$\mathbf{r}_1(\mathbf{u}) = \begin{pmatrix} -1 \\ -C \\ p - Cv \end{pmatrix}, \quad \mathbf{r}_2(\mathbf{u}) = \begin{pmatrix} p_\varepsilon \\ 0 \\ -p_\tau \end{pmatrix}, \quad \mathbf{r}_3(\mathbf{u}) = \begin{pmatrix} -1 \\ C \\ p + Cv \end{pmatrix}$$

We are in situation (ii) of Section 3 with $q_{\rm L} = 1$ and $q_{\rm R} = q = 2$, $m = 1$. One can prove that the dimension of the subspace $E = [\mathbf{r}_{{\rm R},1}, \mathbf{r}_{{\rm R},2}] \cap [\mathbf{r}_{{\rm L},2}, \mathbf{r}_{{\rm L},3}]$ is indeed equal to 1, and applying Theorem 2, that imposing the continuity of the pressure and velocity at the material interface leads to two well-posed linearized boundary value problems.

We pass to the coupling of two *non-linear* gas dynamics systems at a contact discontinuity. Given a left state $\mathbf{u}_\mathrm{L}$, we denote by $\mathbf{S}^1_\mathrm{R}(\mathbf{u}_\mathrm{L})$ the 1–wave curve consisting of states $\mathbf{u}$ which can be connected to $\mathbf{u}_\mathrm{L}$ on the right by either a 1−shock or a 1−rarefaction wave corresponding to the equation of state $p = p_\mathrm{R}(\tau, \varepsilon)$. Similarly, given a right state $\mathbf{u}_\mathrm{R}$, we denote by $\mathbf{S}^3_\mathrm{L}(\mathbf{u}_\mathrm{R})$ the 3-wave curve consisting of states $\mathbf{u}$ which can be connected to $\mathbf{u}_\mathrm{R}$ on the left by a 3−shock or a 3−rarefaction wave corresponding to the equation of state $p = p_\mathrm{L}(\tau, \varepsilon)$. We denote by $\mathscr{S}^1_\mathrm{R}(\mathbf{u}_\mathrm{L})$ (resp. $\mathscr{S}^3_\mathrm{L}(\mathbf{u}_\mathrm{R})$) the projection on the (v, p)-plane of $\mathbf{S}^1_\mathrm{R}(\mathbf{u}_\mathrm{L})$ (resp. $\mathbf{S}^3_\mathrm{L}(\mathbf{u}_\mathrm{R})$).

Proposition 1

Assume that for any pair of states $(\mathbf{u}_\mathrm{L}, \mathbf{u}_\mathrm{R})$ the curves $\mathscr{S}^1_\mathrm{R}(\mathbf{u}_\mathrm{L})$ and $\mathscr{S}^3_\mathrm{L}(\mathbf{u}_\mathrm{R})$ may intersect at one point at most. Then, the coupling conditions (4) are equivalent to

$$v(0+, t) = v(0-, t), \quad p(0+, t) = p(0-, t) \tag{22}$$

Since the curve $\mathscr{S}^1_\mathrm{R}(\mathbf{u}_\mathrm{L})$ (resp. $\mathscr{S}^3_\mathrm{L}(\mathbf{u}_\mathrm{R})$) is tangent to $\mathbf{r}^1_\mathrm{R}(\mathbf{u}_\mathrm{L})$ (resp. $\mathbf{r}^3_\mathrm{L}(\mathbf{u}_\mathrm{R})$, the assumption on the wave curves is indeed satisfied and linked to the assumption for the linear system that $\dim E = 1$. The coupled problem is well-posed.

5. AN EXAMPLE OF A FLUID MODEL IN PLASMA PHYSICS

We consider a classical two-temperature ion–electron plasma model which, after some simplifications, may be written in a conservation form (1), (2) with

$$\mathbf{u} = (\rho, \rho v, \rho s, \rho s_\mathrm{e})^\mathrm{T}, \quad \mathbf{f}_\alpha(\mathbf{u}) = (\rho v, \rho v^2 + p + p_\mathrm{e}, \rho s v, \rho s_\mathrm{e} v_\alpha)^\mathrm{T} \tag{23}$$

ρ, v, p (resp. p_e), are the mass density, mean velocity, pressure of the ion (resp. electron) fluid and $v_\alpha = v_\mathrm{e}$ is the electron velocity which equals v in a zone where the Joule effect is neglected

$$v_\alpha = v_\mathrm{e} = v - \frac{\beta_\alpha}{\rho} \quad \text{where } \beta_\mathrm{L} = 0, \quad \beta_\mathrm{R} = \beta > 0 \tag{24}$$

Thus, only the 4th equations differ on each side of the interface. We also have the closure relations between the pressure and entropy $p = (\rho s)^\gamma$, $p_\mathrm{e} = (\rho s_\mathrm{e})^\gamma$, $\gamma = \frac{5}{3}$. We shall assume that β is constant. The eigenvalues are $v, v_\mathrm{e}, v \pm c$, and the system is indeed hyperbolic, except when $\rho c = \beta$ where $c^2 = \gamma(p + p_\mathrm{e})/\rho$. We have first studied the Riemann problem for $\beta > 0$ (the case $\beta = 0$ is classical). Setting $K_\mathrm{L} = s^\gamma_\mathrm{L} + s^\gamma_\mathrm{e,L}$, $K_1 = s^\gamma_\mathrm{L} + s^\gamma_\mathrm{e,R}$ we can state

Theorem 4

Assume $v_\alpha = v_\mathrm{e} = v - \beta/\rho, \beta > 0$, and $v_\mathrm{e,L} < v_\mathrm{L} - c_\mathrm{L}$. Then under the condition

$$P_\mathrm{L} + \beta^2 \left(\frac{K_\mathrm{L}}{P_\mathrm{L}}\right)^{1/\gamma} \geqslant (\gamma + 1) \left(\beta^2 \frac{K_1^{1/\gamma}}{\gamma}\right)^{(\gamma/\gamma+1)} \quad \text{if } s_\mathrm{e,L} < s_\mathrm{e,R}$$

(no condition if $s_\mathrm{e,L} \geqslant s_\mathrm{e,R}$), the Riemann problem for (23) has a unique 'admissible solution'.

By admissible solution, we mean a solution which depends continuously on the initial data and consisting of constant states separated by contact discontinuities or rarefactions.

Then we study the *coupled Riemann problem* assuming that the wave velocities are ordered following $v_e < v - c < 0 < v < v + c$. We are then in the situation of case (iii) of Section 3.

Theorem 5
Assume the conditions

$$v_e(0-,t) = \left(v - \frac{\beta}{\rho}\right)(0-,t) < (v-c)(0-,t) < 0 < v(0-,t) < (v+c)(0-,t) \tag{25}$$

$$(v-c)(0+,t) < 0 < v_e(0+,t) = v(0+,t) < (v+c)(0+,t) \tag{26}$$

Then the self-similar solutions of the Riemann problem for (1), (2), (4), (23) form a one-parameter family parametrised by $s_e^* \in [0, s_{e,\max}^*]$ where $s_e^* = s_e(0-) = s_e(0+)$ and

$$s_{e,\max}^* = \left\{ \left(\frac{\gamma}{\beta^2}\right)^{\gamma} \left(\frac{1}{\gamma+1}\left(P_L + \beta^2 \left(\frac{K_L}{P_L}\right)^{1/\gamma}\right)\right)^{\gamma+1} - s_L^{\gamma} \right\}^{1/\gamma}$$

For the numerical computations of these solutions, we have tested various methods of upwind type and observed that they converge towards nearly the same solution corresponding to a value of the parameter s_e^* depending on the equation of state, i.e. on the adiabatic exponent γ, and on the initial conditions (we refer to Reference [6] for details). We cannot characterize simply this solution which is not stable with respect to a regularizing process. In the scalar case, however, we have seen [3, corollary of Theorem 7] that, when the initial data u_0 is continuous, the sequence of discrete solutions converges to the unique solution of the coupled problem corresponding to $u_0(0+) = u_0(0-) = u_0(0)$. Numerical experiments in the case of the coupled plasma problem with continuous initial data have lead indeed to realistic results (see Reference [8]).

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